

Time-reversal violation as loop-antiloop symmetry breaking: the Bessel equation, group contraction and dissipation

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Abstract.

We show that the Bessel equation can be cast, by means of suitable transformations, into a system of two damped/amplified parametric oscillator equations. The relation with the group contraction mechanism is analyzed and the breakdown of loop-antiloop symmetry due to group contraction manifests itself as violation of time-reversal symmetry. A preliminary discussion of the relation between some infinite dimensional loop-algebras, such as the Virasoro-like algebra, and the Euclidean algebras $e(2)$ and $e(3)$ is also presented.

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1. Introduction

In this paper we show that the Bessel equation can be cast, by means of suitable transformations, into a system of two parametric oscillator equations, one for a damped oscillator, the other one for an amplified one. We show that the group contraction mechanism [1] is involved in such a relation of the Bessel equation with the dissipation/amplification system.

The interest in such a representation of the Bessel equation is due to the role played by the couple of damped/amplified oscillators in several physical problems [2]. Since Bateman [3] proposed to treat the damped system by doubling its degrees of freedom and introducing the companion (time-reversed) amplified system, the interest in the damped/amplified oscillator system has been growing [4] and its relevance in treating dissipation at classical and quantum level has been stressed [5] - [9]. The couple of damped/amplified oscillators has been fruitfully used to describe inflationary models of the Universe [10], thermal field theories [5], Chern-Simons gauge theory [11], Bloch electrons in metals [11], the dissipative quantum model of brain [12, 13, 14], etc. It also presents features common to the formalism of two-dimension gravity models [15]. In particular, the equivalence, under suitable parametrization, between the spherical Bessel equation and the damped/amplified oscillators was firstly recognized in the study of expanding geometry models [10] and of ordered domain formation in brain models [13].

On the other hand, the possibility of using and of exploiting the properties of special functions in physical problems is *in itself* of great interest, since, as remarked by Wigner [16], "the role which is common to all special functions is to be matrix elements of representations of the simplest Lie groups". There is therefore a strong motivation to analyze the relation between the Bessel equation and the damped/amplified parametric oscillator system also from the perspective of special function theory and group theory. We show that the mechanism of group contraction [1] by which one gets the Euclidean groups $E(2)$ and $E(3)$, whose representations are given in terms of planar and spherical Bessel functions, respectively, introduces the breakdown in the loop-antiloop symmetry around a preferred axis. In turn, this can be read off, in a given re-parametrization, as the breakdown or violation of the time-reversal symmetry.

The analysis in this paper brings us also to consider infinite dimensional loop-algebras, such as the Virasoro-like algebra, in relation with the Euclidean algebras $e(2)$ and $e(3)$. In view of the relevance of infinite dimensional algebraic structures in many physical problems, the preliminary results here presented deserve to be further analyzed, which is our plan for a future work.

The plan of the paper is the following. In Section 2 we consider the Bessel equation for the spherical Bessel functions and derive from it the set of two damped/amplified parametric oscillators. In Section 3 we present similar derivation for the Bessel equation for the planar Bessel functions and we discuss the role played by the group contraction in our derivation. In Section 4 some preliminary results on the relation between the

Virasoro-like algebra and the $e(2)$ and $e(3)$ algebras are presented. Section 5 is devoted to conclusions and further remarks. In the Appendix we show that the Bessel-like equation describes the damped parametric oscillator under suitable conditions.

2. The Bessel equation and dissipation

In this Section we show that the spherical Bessel equation of order n (also called the Bessel equation of fractional order [17]) can be cast, by means of suitable transformations, into a set of two equations representing a couple of damped/amplified parametric oscillators. In the following Section we analyze the case of the planar Bessel equation and the group structure underlying the relation between Bessel equation and damped/amplified oscillators.

The spherical Bessel equation of order n , whose solutions constitute a complete set of (parametric) decaying functions [17], is:

$$\eta^2 J_{n;\eta\eta} + 2\eta J_{n;\eta} + [\eta^2 - n(n+1)] J_n = 0. \quad (1)$$

Here n is an integer or zero number, ($n = 0, \pm 1, \pm 2, \dots$) and, as customary, the labels “; η ” and “; $\eta\eta$ ” denote first and second order derivatives, respectively. As well known, the solutions of Eq. (1), the so called spherical Bessel functions, can be expressed in terms of the first and second kind Bessel functions and their linear combinations (the Hankel functions).

Eq. (1) is invariant under the transformation $n \rightarrow -(n+1)$ and J_n and $J_{-(n+1)}$ are both solutions of the same equation. We can express this by regarding (1) as an eigenvalue equation and saying that J_n and $J_{-(n+1)}$ are degenerate solutions corresponding to the same eigenvalue $n(n+1)$ of the operator $\eta^2 \frac{d^2}{d\eta^2} + 2\eta \frac{d}{d\eta} + \eta^2$.

We now perform in the Eq.(1) the change of variables : $\eta \rightarrow \eta \equiv \epsilon x$ with $x \equiv e^{-t/\alpha}$, where ϵ and α are arbitrary parameters and the new variable t may be thought to denote, e.g., the time variable. By using $w_{n,l} \equiv J_n \cdot (x)^{-l}$, Eq.(1) then goes into the following one:

$$\ddot{w}_{n,l} - \frac{2l+1}{\alpha} \dot{w}_{n,l} + \left[\frac{l(l+1) - n(n+1)}{\alpha^2} + \left(\frac{\epsilon}{\alpha}\right)^2 e^{-2t/\alpha} \right] w_{n,l} = 0, \quad (2)$$

where \dot{w} denotes derivative of w with respect to time t . We now remark that making the choice $l(l+1) = n(n+1)$ the degeneracy between the solutions J_n and $J_{-(n+1)}$ is removed: in other words, a *partition* is induced between the two solution sectors $\{J_n\}$ and $\{J_{-(n+1)}\}$, as shown by the fact that now two *different* sets of equations are obtained, one for $w_{n,l}$ and the other one for $w_{-(n+1),l}$, respectively, each set being composed by two *different* equations, one for $l = -(n+1)$ and the other one for $l = n$. The set for $w_{n,l}$ is

$$\begin{aligned} \ddot{w}_{n,-(n+1)} + \frac{2n+1}{\alpha} \dot{w}_{n,-(n+1)} + \left[\left(\frac{\epsilon}{\alpha}\right)^2 e^{-2t/\alpha}\right] w_{n,-(n+1)} &= 0, \\ \ddot{w}_{n,n} - \frac{2n+1}{\alpha} \dot{w}_{n,n} + \left[\left(\frac{\epsilon}{\alpha}\right)^2 e^{-2t/\alpha}\right] w_{n,n} &= 0, \end{aligned} \quad (3)$$

for $l = -(n+1)$ and for $l = n$, respectively. Similarly, two equations for $w_{-(n+1),l}$ are obtained:

$$\begin{aligned} \ddot{w}_{-(n+1),n} - \frac{2n+1}{\alpha} \dot{w}_{-(n+1),n} + \left[\left(\frac{\epsilon}{\alpha}\right)^2 e^{-2t/\alpha}\right] w_{-(n+1),n} &= 0, \\ \ddot{w}_{-(n+1),-(n+1)} + \frac{2n+1}{\alpha} \dot{w}_{-(n+1),-(n+1)} + \left[\left(\frac{\epsilon}{\alpha}\right)^2 e^{-2t/\alpha}\right] w_{-(n+1),-(n+1)} &= 0, \end{aligned} \quad (4)$$

for $l = n$ and $l = -(n+1)$, respectively. Inspection of Eqs. (3) and Eqs. (4) shows that the symmetry under the transformation $n \rightarrow -(n+1)$ has been broken.

As a further step, let us now choose the arbitrary parameters α and ϵ to be n -dependent: $\alpha \rightarrow \alpha_n$ and $\epsilon \rightarrow \epsilon_n$ (such a choice means that we use $\eta \rightarrow \eta_n \equiv \epsilon_n x_n$ with $x_n \equiv e^{-t/\alpha_n}$). Also, we perform our choice in such a way that $\frac{2n+1}{\alpha_n} \equiv L$ and $\frac{\epsilon_n}{\alpha_n} \equiv \omega_0$ do not depend on n (and on time). By setting $u_n \equiv w_{n,-(n+1)}$, and $v_n \equiv w_{n,n}$, we then see that Eqs. (3) are nothing else than the couple of equations for the damped/amplified parametric oscillators:

$$\begin{aligned} \ddot{u}_n + L \dot{u}_n + \omega_n^2(t) u_n &= 0, \\ \ddot{v}_n - L \dot{v}_n + \omega_n^2(t) v_n &= 0, \end{aligned} \quad (5)$$

with frequency

$$\omega_n(t) = \omega_0 e^{-\frac{Lt}{2n+1}}. \quad (6)$$

Eqs. (5) are sometimes called Hill-type equations [18]. We see that $\omega_n(t)$ approaches to the time-independent value ω_0 for $n \rightarrow \infty$: the frequency time-dependence is thus "graded" by the order n of the original Bessel equation. L and ω_0 , which may be arbitrarily chosen, are characteristic parameters of the oscillator system.

We note that our choice of keeping L independent of n implies that $\alpha_{-(n+1)} = -\alpha_n$. Then the transformation $n \rightarrow -(n+1)$ leads to solutions (corresponding to $J_{-(n+1)}$) which have frequencies exponentially increasing in time (cf. Eq. (6)). These solutions can be respectively obtained from the ones of Eqs. (5) by time-reversal $t \rightarrow -t$ and exchanging u with v (which we refer to as "charge conjugation"). In the large n limit ($\omega_n \rightarrow \omega_0$) u_n and v_n are each the time-reversed of the other one and in that limit the two sectors $\{J_i\}$, $i = n, -(n+1)$ are mapped one into the other one.

We thus finally recognize the core of the relation between the spherical Bessel equation and the dissipation/amplification phenomenon: the breakdown of the $n \rightarrow -(n+1)$ symmetry of the original spherical Bessel equation corresponds in our representation to the breakdown of time-reversal symmetry (the emergence of the *arrow of time*) in the manifold of the solutions (the spherical Bessel functions) $\{J_i\}$, $i = n, -(n+1)$.

In the following Section we will see that such a feature has its root in the structure of the $E(3)$ group, whose representations, as well known, can be indeed constructed by means of the spherical Bessel functions. We will also see that the breakdown of the $n \rightarrow -(n+1)$ symmetry can be viewed as the breakdown of the loop-antiloop symmetry around a preferred direction in the 4D-space.

We observe that the functions $w_{n,n}$ and $w_{n,-(n+1)}$ are "harmonically conjugate" functions in the sense that they may be re-conducted to the single parametric oscillator

$$\ddot{r}_n + \Omega_n^2(t) r_n = 0, \quad (7)$$

where $r_n(t)$ is defined through $w_{n,-(n+1)}(t) = \frac{1}{\sqrt{2}} r_n(t) e^{\frac{-Lt}{2}}$ and $w_{n,n}(t) = \frac{1}{\sqrt{2}} r_n(t) e^{\frac{Lt}{2}}$. Ω_n is the common frequency:

$$\Omega_n(t) = \left[\left(\omega_n^2(t) - \frac{L^2}{4} \right) \right]^{\frac{1}{2}}. \quad (8)$$

Remarkably, the first of Eqs. (5), with $n = 1$ is commonly used in expanding geometry (inflationary) models of the Universe [10]. In that case L denotes the Hubble constant. For a discussion of expanding geometry models in terms of Eqs. (5) see [10].

In conclusion, the spherical Bessel equation represents, under suitable transformations, a two-fold hierarchy of couples of parametric oscillators with constant damping/amplification given by L and with time dependent frequency graded by n and by $-(n+1)$. Transition from one tower to the other one ($n \rightarrow -(n+1)$) is induced by time-reversal $t \rightarrow -t$ combined with "charge conjugation" $u \rightarrow v$.

In the Appendix we show that the Bessel-like equation

$$J'' + \frac{\alpha}{\eta} J' + \left(1 - \frac{\beta^2}{\eta^2} \right) J = 0, \quad (9)$$

where $\alpha = 1$ or 2 , β is an arbitrary real constant and $'$ indicates the derivative with respect to the independent variable η , may represent parametric oscillators with constant damping and with different functional choices for the frequency. Vice-versa, from the equation of the damped parametric oscillator, by use of convenient transformations, one can always obtain the Bessel-like equation (9).

In the next Section we consider the group structure underlying the planar and the spherical Bessel equation in order to clarify their connection with the damped/amplified parametric oscillator system.

3. Group contraction and Bessel equations

It is well known that the representations of the Euclidean groups can be constructed in terms of the Bessel functions [16, 17]. We are therefore interested in these groups in the present Section. Features analyzed in the previous Section will be found to be rooted in these groups.

We will focus our attention on $E(2)$ and on $E(3)$, the Euclidean groups in the plane and in the space, respectively. Their relation with the Laplace equations in three and four dimensions will be summarized (see [16] for details) in view of the connection with Bessel equations.

3.1. $E(2)$ and planar Bessel equations

$E(2)$ is the group of the $T(\vec{v})R(\theta)$ transformations, where $T(\vec{v})$ is the translation in the plane by the vector \vec{v} ($\vec{v} \equiv (a, b)$) and $R(\theta)$ is the rotation of the plane around the origin by the angle θ . The associated Lie algebra is given in terms of the two translation generators P_a, P_b and of the rotation generator M :

$$[P_a, P_b] = 0, \quad [P_a, M] = -P_b, \quad [P_b, M] = P_a. \quad (10)$$

The invariant operator of $E(2)$ is $P^2 = P_a^2 + P_b^2 = P_+ P_- = P_- P_+$, with $P_{\pm} \equiv P_a \pm iP_b$, which has non-positive eigenvalue, $-p^2$.

The group transformations on square integrable functions f are represented by the action of the operator $D(\vec{v}, \theta)$ (see e.g. [16]):

$$\begin{aligned} D(\vec{v}, 0)f(\phi) &= e^{i\vec{p}\cdot\vec{v}}f(\phi), \\ D(0, \theta)f(\phi) &= f(\phi - \theta). \end{aligned} \quad (11)$$

In particular, it is possible to assume as basis functions the complete set of normalized eigenfunctions of the rotation subgroup $\{f_n(\phi)\}$, $f_n \equiv (2\pi)^{-\frac{1}{2}} i^{-n} e^{in\phi}$: $D(0, \theta)f_n(\phi) = e^{in\theta}f_n(\phi)$. The complete representation of $E(2)$ can be expressed in the form:

$$\begin{aligned} D(\vec{v}, \theta)f_n &= \sum_m \Delta(\vec{v}, \theta)_{mn} f_m, \\ \Delta(\vec{v}, \theta)_{mn} &= (-1)^{m-n} e^{-im\beta} J_{m-n}(pr) e^{in(\beta-\theta)}, \end{aligned} \quad (12)$$

where (r, β) are the polar coordinates of \vec{v} and J_m is the planar Bessel function of order m . Eqs. (12) provide the representation of $E(2)$ in terms of the Bessel functions J_m [16].

Now we want to study the P^2 eigenvalue equation. To this aim we consider the 3D-Laplace equation:

$$\nabla^2 \psi \equiv \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) \psi = 0, \quad (13)$$

which refers to an isotropic and homogeneous 3D-space. By choosing, instead of the spherical or rectangular coordinates, the cylindrical ones, we have:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{r^2 \partial \theta^2} + \frac{\partial^2}{\partial x_3^2} \right) \psi = 0, \quad (14)$$

and we search for solutions of the type:

$$\psi(r, \theta, x_3) = \varphi(r, \theta) \cdot \sigma(x_3), \quad \frac{\partial^2}{\partial x_3^2} \sigma(x_3) \equiv p^2 \sigma(x_3). \quad (15)$$

Square integrable eigenfunctions (for p positive) are obtained by selecting, for positive x_3 , the solution: $\sigma = e^{-x_3 p}$ and for negative x_3 the solution: $\sigma = e^{x_3 p}$.

It is clear that the choice of the cylindrical, instead of the spherical coordinates, breaks the symmetry of 3D-spatial rotation group $SO(3)$: indeed, when cylindrical coordinates are chosen x_3 is differently treated with respect to the two remaining

coordinates and this singles out a privileged axis for rotations. The resulting symmetry group is $E(2)$, the group contraction of $SO(3)$. This is manifest in the fact that the Laplace equation (14) reduces to the eigenvalue equation for P^2 (realized in polar coordinates), i.e. the 2D-Helmholtz equation:

$$P_+ P_- \varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{r^2 \partial \theta^2} \right) \varphi = -p^2 \varphi. \quad (16)$$

If we solve this equation by assuming $\varphi(r, \theta) = f(r) \cdot e^{in\theta}$, we obtain $f(r) = J_n(pr)$, being $J_n(pr)$ the solution of the planar Bessel equation of order n :

$$J_{n;\eta\eta} + \frac{1}{\eta} J_{n;\eta} + \left[1 - \frac{n^2}{\eta^2}\right] J_n = 0, \quad (17)$$

where $\eta = pr$. Positive/negative values of n correspond to positive/negative rotations (loop/antiloop) around the x_3 axis, i.e., they correspond to different orientations of the x_3 axis. The related solutions are, therefore, different [17]. This occurrence (and the similar one in the case of spherical Bessel functions studied below in subsection 3.2) bring us in a natural way to consider topological properties of Bessel functions related with loop operators and loop-algebras, on which we comment in Section 4. Notice that, in spite of the fact that the solutions are not symmetric under the reversal of the x_3 axis (i.e. under time-reversal in our choice $x_3 \equiv t$) Eq. (17) is invariant under the $n \rightarrow -n$ exchange. The $n \rightarrow -n$ invariance of Eq. (17) actually reflects the existence of the two sets of the $SO(3)$ independent representations: the D^n and the D^{-n-1} [19].

Now, by using a procedure similar to the one followed in Section 2 for the spherical Bessel equation (1), by setting $w_{n,l}(\eta) = (\eta)^{-l} J_n(\eta)$, $\eta = \epsilon e^{-t/\alpha}$, and by introducing the mirror parameter $l = \pm n$, the planar Bessel equation (17) gives the couple of damped/amplified harmonic oscillators:

$$\begin{aligned} l = -n : \quad \ddot{w}_{n,-n} + \frac{2n}{\alpha} \dot{w}_{n,-n} + \left[\left(\frac{\epsilon}{\alpha}\right)^2 e^{-2t/\alpha} \right] w_{n,-n} &= 0, \\ l = n : \quad \ddot{w}_{n,n} - \frac{2n}{\alpha} \dot{w}_{n,n} + \left[\left(\frac{\epsilon}{\alpha}\right)^2 e^{-2t/\alpha} \right] w_{n,n} &= 0, \end{aligned} \quad (18)$$

where the value of l is connected with the choice of negative/positive eigenvalues of the rotation generator M .

Summarizing, we have shown that the breakdown of the rotational symmetry of $SO(3)$ which leads to its group contraction $E(2)$ introduces a crucial difference (loss of loop-antiloop symmetry, indeed) in the double choice of the x_3 axis orientation. This, in turn, results in the difference between the planar Bessel functions of order $+n$ and the ones of order $-n$, in terms of which the $E(2)$ representations can be built. Then, we have shown that the planar Bessel equation (17) can be cast, by a convenient reparametrization, into the set of eqs. (18) for the damped/amplified harmonic oscillators: the *mirror* index $\pm n$ of the Bessel functions is thus associated to the *couple* of damped/amplified harmonic oscillators (it is a time-mirror index).

It is also worth to recall that the contraction of $SO(3)$ to $E(2)$ can be geometrically depicted as the projection of the S_2 sphere (corresponding to $SO(3)$) on the plane (e.g.

the plane tangent to one of the poles of the sphere). It is then clear that the radius, say ρ , of the sphere acts as a "scale" (one may introduce $J_x \equiv \rho \cdot P_a$, $J_y \equiv \rho \cdot P_b$, $J_z \equiv M$ with the J s denoting the generators of the algebra $so(3)$): the $E(2)$ translations in the tangent plane are "good" approximations in the limit $\rho \rightarrow \infty$, namely for distances much smaller than ρ . The physical meaning of this is that the $SO(3)$ contraction to $E(2)$ manifests itself in local observations. However, in the local observation process the x_3 axis orientation is "locked". Which amounts to the loss of symmetry of the solutions under the $n \rightarrow -n$ exchange (breakdown of the loop-antiloop symmetry). As a matter of fact, specifying the direction of the x_3 axis, i.e. choosing one of the two possible forms for σ (cf. Eq. (15)), produces topologically inequivalent configurations [20] (see also Section 4).

We also observe that, identifying x_3 with the time t coordinate, p has the dimensions of an energy over an action (cf., e.g., the eigenfunctions $\sigma = e^{\pm x_3 p}$).

As a final comment, we note that the so called "spinor" representation of $SO(3)$, which involves half-integer values of n (corresponding to represent $SO(3)$ in terms of the Pauli matrices $\sigma_i, i = 1, 2, 3$), maps the planar Bessel equation into the harmonic oscillator equation. In fact, for $n = \pm \frac{1}{2}$ and with $w \equiv \sqrt{\eta} J_{\pm \frac{1}{2}}$, Eq.(17) reduces to the harmonic oscillator equation with frequency p (r plays the role of time):

$$\frac{d^2 w(r)}{dr^2} + p^2 w(r) = 0, \quad r = \frac{\eta}{p}. \quad (19)$$

In this connection it has been observed [21] that the oscillator so obtained may be thought as a possible classical analogue of a Fermi oscillator based on a 'rotation system'.

3.2. $E(3)$ and the spherical Bessel equation

The analysis presented in the previous subsection can be extended to $E(3)$, the Euclidean group in the space, which is the group contraction of $SO(4)$. The algebra $e(3)$ has six generators P_i and $M_i, i = 1, 2, 3$, corresponding to the translation and to the rotation generators, respectively. The commutation relations are:

$$[P_i, P_j] = 0, \quad [M_i, M_j] = \epsilon_{ijk} M_k, \quad [P_i, M_j] = \epsilon_{ijk} P_k; \quad (20)$$

We observe that in the contraction process the $SO(3)$ subgroup generated by the M_i 's is left unchanged. The algebra $e(3)$ has two invariants, $P^2 = \Sigma P_i^2$ and $\Sigma P_i \cdot M_i$.

In the 4D-space the Laplace equation for the function $\psi = \psi(x_1, x_2, x_3, x_4)$ may be solved by using the position: $\psi = \varphi(r, \theta, \phi) \cdot \sigma(x_4)$, (r, θ, ϕ spherical coordinates, $\sigma = e^{\pm x_4 p}$). By a procedure similar to the one of the previous subsection, x_4 may be considered to play the role of time t . The resulting equation (corresponding to (16)) is solved by the function $\varphi = Y_{n,m}(\theta, \phi) \cdot J_n(pr)$ where $Y_{n,m}$ is the spherical harmonics and J_n is the solution of the spherical Bessel equation:

$$J_{n;\eta\eta} + \frac{2}{\eta} J_{n;\eta} + \left[1 - \frac{n(n+1)}{\eta^2}\right] J_n = 0, \quad (21)$$

As in the previous $E(2)$ case for the planar Bessel functions, the spherical Bessel functions depend on the continuous eigenvalue p^2 of P^2 and are labelled by n which is related with the discrete eigenvalue $n(n+1)$ of the rotation operator \mathbf{M}^2 .

The order n (integer or zero) classifies the representations D^n of the compact subgroup $SO(3)$ of $E(3)$. Actually, as already said for the planar case, the existence of two sets of $SO(3)$ independent representations (the D^n and the D^{-n-1}) [19] is reflected in Eq.(21) through its invariance under the transformation $n \rightarrow -(n+1)$. For $n = 0$, or $n = -1$ equation (21) reduces to the harmonic oscillator equation with frequency p :

$$\frac{d^2 w(r)}{dr^2} + p^2 w(r) = 0, \quad r = \frac{\eta}{p}, \quad (22)$$

where $w = \eta J_0$ or ηJ_{-1} . The harmonic oscillator thus appears to be related to the “ground state” (in the D^n or D^{-n-1} spectra) of the Bessel system. Here, of course, the so-called “true” representation of $SO(3)$, i.e., the one with integer values of n [19], has been used.

Again, the breakdown of the symmetry under the transformation $n \rightarrow -(n+1)$ is built in the geometrical structure of the $E(3)$ group: the breakdown of the $n \rightarrow -(n+1)$ symmetry is nothing but the breakdown of the x_4 axis reversal symmetry (breakdown of the loop-antiloop symmetry), i.e. of time-reversal symmetry when x_4 is considered to be the time variable, which brings us back to the analysis of subsection 3.1 (see also Section 2). Also in the present case, the $SO(4)$ contraction to $E(3)$ manifests itself in local observations and the x_4 axis orientation then gets “locked”. Which amounts to the loss of symmetry of the solutions under the $n \rightarrow -(n+1)$ exchange.

4. Loop algebras as extension of Euclidean algebras

Infinite-dimensional algebras and in particular the so-called “loop-algebras” are particularly interesting and useful algebraic structures. It is well known that they can be constructed on some finite-dimensional group as $SU(1,1)$ [22] and many applications exist which exploit such a feature.

In view of the strict relation between the Euclidean groups, the Bessel functions and the dissipation/amplification processes we have discussed in the previous Sections, it is also much interesting to consider some topological properties of Bessel functions in connection with loop-algebras.

It is known [20] that the Bessel functions describe solutions with different Pontryagin number in the punctured plane $R^2/(0)$, where the elements of the homotopy group, Π_n , may be represented by differential operators acting on analytic functions:

$$\Pi_n \equiv \frac{\partial^n}{\partial z^n}, \quad n \in N, \quad (23)$$

with $\Pi_n \cdot \Pi_m = \Pi_{n+m}$ and n is the loop number around the hole. It is possible to distinguish two different kinds of behavior, corresponding to two different functions:

$$\frac{\partial^n}{z \partial z^n} \varphi_m(z) = (-)^m \varphi_{m+n}(z), \quad \varphi_m(z) = \frac{J_m(z)}{z^m} \quad (24)$$

and

$$\frac{\partial^n}{z \partial z^n} \psi_m(z) = \psi_{m-n}(z), \quad \psi_m(z) = z^m J_m(z), \quad (25)$$

so that on φ , Π_n acts in counter-clockwise way while on ψ it acts in clockwise way. $J_m(z)$ is the planar Bessel function (Bessel function of integer order).

Eqs.(24) and (25) are the well known differential formulae for the planar Bessel functions [17]; analogous formulae are true for the functions

$$\varphi_m(z) = j_m(z) z^{-m}, \quad \psi_m(z) = z^{(m+1)} j_m(z), \quad (26)$$

where the j_m are the spherical Bessel functions [20].

Notice that while for the planar Bessel functions, the “raising” and “lowering” functions coincide for $m = 0$ (no loops), this is not true with the spherical Bessel functions.

These topological properties, in the planar and in the spherical case, of the φ and ψ functions readily remind us of the x_3 - and x_4 -reversal symmetry breakdown discussed in Section 3.

Thus, in view of the topological properties of the Bessel functions and, on the other hand, of their relation with the Euclidean groups, it is remarkable that the algebra $e(3)$ can be related to a particular structure of loop-algebras. This goes as follows.

Let us focus our attention in particular on the Virasoro algebra which plays a central role in the conformal field theories. We will show that in analogy with the rotation algebras, it is possible to follow a contraction procedure which maps the Virasoro algebra into a sort of generalization of the Euclidean algebra $e(3)$. This result derives from a general construction based on the so called graded contraction method [23, 24].

The commutation relations of the Virasoro algebra \mathcal{L} of central charge c (c commuting with all the T 's) are

$$[T_n, T_m] = (n - m)T_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \quad m, n \in \mathbf{Z}. \quad (27)$$

The \mathbf{Z}_2 -grading of the algebra consists in dividing the set of the T_n generators into an even set $L_0 \equiv \{A_n, c\}$ and an odd set $L_1 \equiv \{B_n\}$, with

$$A_n = \frac{1}{2} \left(T_{2n} + \frac{c}{8} \delta_{n,0} \right), \quad B_n = \frac{1}{2} T_{2n+1}, \quad (28)$$

so that $\mathcal{L} = L_0 \oplus L_1$ and

$$[L_0, L_0] \subseteq L_0, \quad [L_0, L_1] \subseteq L_1, \quad [L_1, L_1] \subseteq L_0. \quad (29)$$

The commutation relations of the graded generators are explicitly given by [25]

$$[A_n, A_m] = (n - m)A_{n+m} + \frac{2c}{12}(n^3 - n)\delta_{n+m,0}, \quad (30)$$

$$[B_n, B_m] = (n - m)A_{n+m+1} + \frac{2c}{12}\left(n - \frac{1}{2}\right)\left(n + \frac{1}{2}\right)\left(n + \frac{3}{2}\right)\delta_{n+m+1,0}, \quad (31)$$

$$[A_n, B_m] = \left(n - m - \frac{1}{2}\right)B_{n+m}. \quad (32)$$

Eq. (30) shows that $\{A_n, c\}$ is again a Virasoro algebra but with central charge $2c$.

We can then consider the \mathbf{Z}_2 -graded contraction of the algebra (30) -(35) [25]:

$$[A_n, A_m] = (n - m)A_{n+m} + \frac{2c}{12}(n^3 - n)\delta_{n+m,0} , \quad (33)$$

$$[B_n, B_m] = 0 , \quad (34)$$

$$[A_n, B_m] = (n - m - \frac{1}{2})B_{n+m} . \quad (35)$$

Our remark is now that, in the centerless case ($c = 0$), the A_0 and $A_{\pm 1}$ generators close the algebra isomorphic to $so(3) \sim su(2)$ and that the set of these three generators and the operators $B_{-\frac{1}{2}}$, $B_{\frac{1}{2}}$ and $B_{-\frac{3}{2}}$ close the $e(3)$ isomorphic algebra. This is shown by setting:

$$\begin{aligned} M_+ &\equiv A_1 , & M_- &\equiv A_{-1} , & M_3 &\equiv iA_0 , \\ P_+ &\equiv B_{\frac{1}{2}} , & P_- &\equiv B_{-\frac{3}{2}} , & P_3 &\equiv iB_{-\frac{1}{2}} , \end{aligned} \quad (36)$$

where the M s and P s generators satisfy the commutation relations (20).

This result has a general extension, i.e. the algebra $\mathcal{E}_n \equiv \{A_0, A_{\pm n}\} \oplus \{B_{-\frac{1}{2}}, B_{\pm n - \frac{1}{2}}\}$ reproduces the $e(3)$ algebra for each integer value of n , provided the following positions are assumed:

$$\begin{aligned} M_+ &\equiv \frac{1}{n}A_n , & M_- &\equiv \frac{1}{n}A_{-n} , & M_3 &\equiv \frac{i}{n}A_0 , \\ P_+ &\equiv B_{n - \frac{1}{2}} , & P_- &\equiv B_{-n - \frac{1}{2}} , & P_3 &\equiv iB_{-\frac{1}{2}} . \end{aligned} \quad (37)$$

As final remark we notice that the $e(2)$ -algebra can be obtained as a subalgebra of (37) by choosing $A_{\pm n} = 0$, for non-zero values of n .

Our conclusion is that the extension of the Virasoro algebra by means of its \mathbf{Z}_2 -grading with the subsequent step of the \mathbf{Z}_2 -graded contraction appears as a n -graded hierarchy of Euclidean algebras. In view of the results obtained in the previous Sections, we see that an interesting relation emerges between the couple of damped/amplified parametric oscillators graded by n and the loop algebras considered in the present Section.

Exhibiting the link between these mathematical structures appears in itself interesting. However, work is still needed in order to fully recognize its physical meaning. Further study in such a direction is in our plans.

5. Final remarks and conclusions

In this paper we have discussed the relation between the (spherical as well as the planar) Bessel equation and the dissipation/amplification processes. Specifically, we have shown that, after suitable re-parametrization, the breakdown of the $n \rightarrow -(n+1)$ ($n \rightarrow -n$) symmetry of the original spherical (planar) Bessel equation corresponds to the breakdown of time-reversal symmetry (the emergence of the *arrow of time*) in the manifold of the solution (the Bessel functions) $\{J_i\}$, $i = n, -(n+1)$ ($i = n, -n$). We have

also shown that the mathematical structure through which such a breakdown is realized is the mechanism of group contraction leading to the Euclidean groups $E(3)$ and $E(2)$, whose representations, as well known, can be constructed by means of the spherical and the planar Bessel functions, respectively. For the spherical case, the $n \rightarrow -(n+1)$ symmetry of the Bessel equation reflects the degeneracy, for each n , between the two independent representations of $SO(3)$, D^n and D^{-n-1} . The role of the contraction mechanism is the one of removing such a degeneracy. Then, the equivalence between the two representations D^n and D^{-n-1} is broken and different parametric oscillators correspond to each of them. In the sector $\{i = n\}$ ($\{i = -(n+1)\}$), solutions with $l = -(n+1)$ and with $l = n$ correspond to damped or amplified (amplified or damped) solutions, respectively. In the large n limit ($\omega_n \rightarrow \omega_0$) these solutions are each the time-reversed of the other one and in that limit the two sectors $\{J_i\}$, $i = n, -(n+1)$ are mapped one into the other. We have seen that, *mutata mutandis*, similar considerations also hold for the planar case. The bottom of the representations ($n = 0$, or $n = -1$ and in the spinor representation $n = \pm \frac{1}{2}$) corresponds to the undamped/un-amplified harmonic oscillator.

Summarizing, we can rephrase our finding in the following way: the effect of the group contraction (removal of the $D^n - D^{-n-1}$ degeneracy) is to “split” the original Bessel equation, conveniently reparametrized, into a couple of damped/amplified oscillators for each n .

We also observe that the fact that the symmetry of the Bessel equation is not the symmetry of the solutions reminds us of the phenomenon of the spontaneous breakdown of symmetry in Quantum Field Theory (QFT) which occurs when the *continuous* symmetry of the dynamical equations is not the symmetry of the physical vacuum. In the Bessel equation case the *discrete* time-reversal symmetry is broken. However, since time-reversal symmetry breakdown manifests itself in the appearing of dissipation/amplification phenomena, also continuous time translational symmetry is broken (energy non-conservation for the damped (or amplified) system), making the resemblance with the QFT case stronger. One further point of contact with the quantum case is in the fact that in QFT the effects of spontaneous breakdown of symmetry are solely observable in the contraction limit, namely at the observation scale (“locality”) which is always “small” with respect to the system volume (the infinite volume limit) [26]. Also in the present case time-reversal symmetry is broken at local, observational level and there dissipation/amplification phenomena become observable phenomena.

It is interesting to observe that, as well known [16], the Weyl-Heisenberg algebra of the quantization procedure is also obtained through the group contraction of the $SO(3)$ group (or, in other instances [27], of the $SU(1,1)$ group).

In this paper our analysis has been limited to the classical sector. The problem of quantization of the damped harmonic oscillator, a prototype of dissipative system, has been extensively studied in a number of papers [2, 4], [5] - [11], [13]. We recall that in order to perform the canonical quantization of the simple damped harmonic oscillator, one “doubles” the system by introducing the time-reversed copy of it (the

companion amplified oscillator) [4, 5]. Canonical quantization is in fact only possible for Hamiltonian systems, and the doubling of the dissipative (non-hamiltonian, open) system amounts to "closing" it by inclusion of the environment (represented by the doubled, amplified oscillator, indeed). At a classical level, however, it is possible to solve the damped oscillator equation and, in principle, to treat the dissipative system by ignoring the environment (i.e. there is no necessity of introducing the amplified oscillator as instead required in the canonical quantization procedure). Remarkably, from our discussion in the present paper it emerges that Bessel equation may represent both the damped and the amplified oscillator as an inseparable "double-face" unity also at the classical level.

Finally, we have also considered some topological properties of the Bessel functions in connection with Virasoro-like loop-algebras and we have presented some preliminary results on the relation between such algebras and the algebras $e(2)$ and $e(3)$. More work is still necessary in this direction in view of the physical relevance of infinite dimensional algebras.

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Appendix A. Bessel equation vs damped parametric oscillators

We show in the following that, under several choices of suitable transformations, the Bessel-like equation

$$J'' + \frac{\alpha}{\eta} J' + \left(1 - \frac{\beta^2}{\eta^2}\right) J = 0 \quad (\text{A.1})$$

describes damped parametric oscillators with constant damping. In Eq.(A.1) $\alpha = 1$ or 2 , β is an arbitrary real constant and $'$ indicates the derivative with respect to the independent variable η .

By using the position:

$$J(\eta) = x(t) f(\eta), \quad t = t(\eta), \quad (\text{A.2})$$

Eq.(A.1) turns into:

$$f(t')^2 \left\{ \ddot{x} + \dot{x} \left[2 \frac{f'}{f} \frac{1}{t'} + \frac{t''}{(t')^2} + \frac{\alpha}{\eta t'} \right] + x \left[\frac{f''}{f} + \frac{\alpha f'}{\eta f} + 1 - \frac{\beta^2}{\eta^2} \right] \frac{1}{(t')^2} \right\} = 0. \quad (\text{A.3})$$

Eq.(A.3) describes a damped parametric oscillator, with constant damping γ and frequency ω , provided the following two conditions are satisfied:

$$2 \frac{f'}{f} \frac{1}{t'} + \frac{t''}{(t')^2} + \frac{\alpha}{\eta t'} = \gamma, \quad \gamma \text{ constant}, \quad (\text{A.4})$$

i.e.

$$f = f_0 e^{(\gamma/2)t} \eta^{-\alpha/2} (t')^{-\frac{1}{2}}, \quad (\text{A.5})$$

and

$$\left[\frac{f''}{f} + \frac{\alpha}{\eta} \frac{f'}{f} + 1 - \frac{\beta^2}{\eta^2} \right] \frac{1}{(t')^2} = \omega^2. \quad (\text{A.6})$$

Let us consider conditions (A.4)–(A.6) for specific choices for $t(\eta)$:

$$\text{A)} \quad t = \eta^l, \quad l \text{ integer}$$

$$f \propto \exp\left(\frac{\gamma\eta^l}{2}\right) \eta^{-\frac{\alpha}{2} + \frac{(1-l)}{2}}. \quad (\text{A.7})$$

In the simple case $l = 1$ ($t = \eta$), ω is given by

$$\omega^2 = \left(\frac{\alpha}{2} - \frac{\alpha^2}{4} - \beta^2\right) \frac{1}{t^2} + \left(\frac{\gamma^2}{4} + 1\right). \quad (\text{A.8})$$

Note that for $\frac{\alpha}{2} - \frac{\alpha^2}{4} = \beta^2$, ω^2 gets constant values.

$$\text{B)} \quad t = \exp(q\eta), \quad q \text{ integer}$$

$$f \propto \eta^{-\alpha/2} \exp\left[-\frac{q\eta}{2} + \frac{\gamma}{2}e^{(q\eta)}\right], \quad (\text{A.9})$$

with

$$\omega^2 = \left(\frac{\gamma}{2}\right)^2 + \left(\frac{1}{qt\eta}\right)^2 \left[\frac{\alpha}{2}\left(1 - \frac{\alpha}{2}\right) - \beta^2\right] + \frac{1}{t^2} \left[\frac{1}{4} + \frac{1}{q^2}\right], \quad (\text{A.10})$$

that is, for $\alpha = 1$ and $\beta = \frac{1}{2}$:

$$\omega^2 = \left(\frac{\gamma}{2}\right)^2 + \left[\frac{1}{4} + \frac{1}{q^2}\right] \frac{1}{t^2}. \quad (\text{A.11})$$

$$\text{C)} \quad t = \ln(q\eta), \quad q \text{ integer}$$

$$f \propto \eta^{(\gamma+1-\alpha)/2}, \quad (\text{A.12})$$

and

$$\omega^2 \propto \mathcal{A} + \mathcal{B}e^{2t}, \quad \mathcal{A}, \mathcal{B} \text{ constant}. \quad (\text{A.13})$$

$$\text{D)} \quad t = \sin(q\eta), \quad q \text{ integer}$$

$$f \propto \exp\frac{\gamma \sin q\eta}{2} \eta^{-\alpha/2} (\cos q\eta)^{-\frac{1}{2}}, \quad (\text{A.14})$$

and, for $\alpha = 1, \beta = \frac{1}{2}$,

$$\omega^2 \propto \mathcal{A} + \frac{\mathcal{B}}{(\cos q\eta)^2} + \frac{\mathcal{C}}{(\cos q\eta)^4}. \quad (\text{A.15})$$

It is straightforward that (almost) all the forms for ω are possible. Vice-versa, under a suitable transformation, it is always possible to turn the equation for a parametric oscillator, with constant damping γ and frequency ω , into a Bessel-like equation of type (A.1).

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